Calculation of Shock Front Parameters in a Plasma

B. Ahlborn and M. Salvat

Institut für Plasmaphysik, Garching bei München
and Department of Physics, University of British Columbia

Introducing an effective ratio of the specific heats, exact values for the shock front parameters can be obtained, if thermal equilibrium may be assumed. Exact values for the shock front parameters in a singly ionized non-equilibrium plasma can be obtained only if one of the end values is measured.

It is well known how to calculate the parameters (pressure p_2, density ϱ_2, temperature T_2, and velocity v_2) in the shock heated zone of a gas flow if the specific heats remain constant [1].

These calculations, however, become difficult in a plasma flow, where the specific heats (and thus $\gamma = c_p/c_v$) vary [2-4]. In order to use the convenient formalism of gas dynamics some authors [5-7] introduced an effective adiabatic exponent, which we will call g. Lunkin [5] (1959) showed, that $g = h/u$, where h is the enthalpy and u the internal energy per gram. g depends on pressure and temperature but, as the variations are only small, it is practice [7], to estimate g so that one can obtain appropriate values of the shock front parameters in a plasma.

In this paper it is shown, that g can also be used to provide the exact p_2, T_2, v_2, ϱ_2, h_2 in the shock heated plasma, and these quantities can be obtained as tabulated functions of the Mach number, without any approximations or recursions, provided that thermal equilibrium may be assumed.

In the absence of thermal equilibrium, the end values can be calculated if any one of them is measured. In a non thermal plasma we calculate the ion density n_i rather then the temperature, which is no longer uniquely defined. Deviations from thermal equilibrium are reflected in a change of the magnitude of g.

General Shock Equations

The shock front parameter p_2, ϱ_2, ... produced by a strong shock in a gas flow are in principle determined by the equations of conservation of mass momentum and energy (the frame of reference moves with the shock front)

$$\varrho_1 v_1 = \varrho_2 v_2 , \tag{1}$$

$$p_1 + \varrho_1 v_1{}^2 = p_2 + \varrho_2 v_2{}^2, \tag{2}$$

$$\tfrac{1}{2} v_1{}^2 + h_1 = \tfrac{1}{2} v_2{}^2 + h_2 \tag{3}$$

and by the two conditions of state $\varrho = \varrho(p, T)$ and $h = h(p, T)$ if the initial values p_1, ϱ_1, v_1, ... are specified. The system can be solved analytically in the case of a non reacting gas where h and ϱ are simple functions of p and T. In a plasma in thermal equilibrium however h and ϱ depend [8] on the partial pressures of ions, electrons and neutrals which are governed by the Saha equation. Though it is always possible to obtain numerical data for ϱ and h, simple analytical equations for enthalpy and density as functions of pressure and temperature can no longer be obtained. Thus it is not possible to solve analytically for the end values in the shock heated zone of a plasma flow. However one can obtain quasi analytical solutions introducing

$$g = h/u \tag{4}$$

to replace the strongly pressure and temperature dependent enthalpy. Fig. 1, 2, 3 show $g(p, T)$ for argon, helium and hydrogen in thermal equilibrium. (It should be noted, that g is also defined in a non thermal plasma — see below.) With the thermodynamical relationship $u = h - p/\varrho$ we obtain from Eq. (4)

$$h_2 = [g_2/(g_2 - 1)] \cdot p_2/\varrho_2 . \tag{5}$$

[1] L. Landau and Lifschitz, Fluid Mechanics, Pergamon Press, New York 1959.
[2] W. Fuchs and J. Artmann, Z. Physik **172**, 118 [1963].
[3] W. Ohlendorf, Institut für Plasmaphysik Garching, Internal Report IPP 3/8.
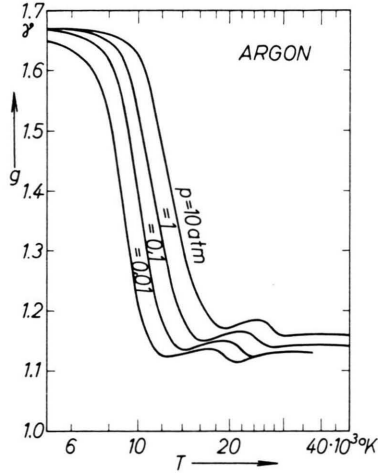[4] W. Böttcher and H. Dammann, Z. Naturforschg. **18 a**, 580 [1963].
[5] Y. P. Lunkin, Soviet Phys.-Techn. Phys. **4**, 155 [1959].
[6] B. Ahlborn and M. Salvat, Institut für Plasmaphysik Garching, Internal Report IPP 3/2 [1962].
[7] W. Böttcher, Z. Naturforschg. **20 a**, 1262 [1965].
[8] W. Finkelnburg and H. Maecker, Handbuch der Physik, Vol. 22, Verlag Springer, Berlin–Göttingen–Heidelberg 1956.

Fig. 1. $g(p, T)$ for argon plasma in thermal equilibrium [10].
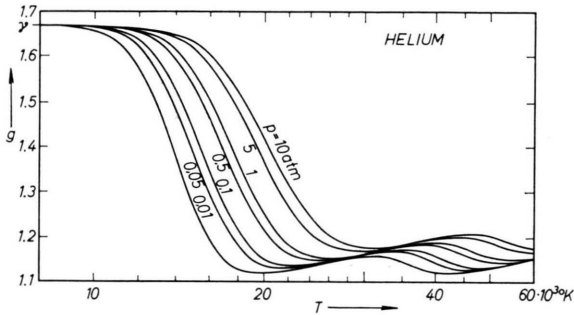
Fig. 2. $g(p, T)$ for helium plasma in thermal equilibrium [13].
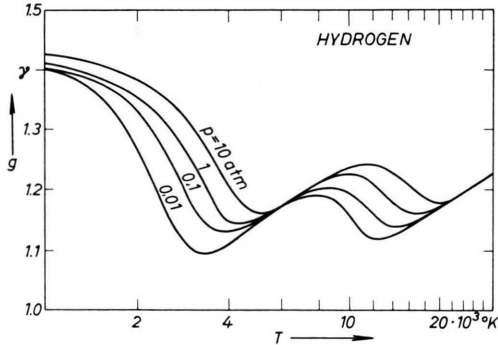
Fig. 3. $g(p, T)$ for hydrogen plasma in thermal equilibrium [7].

This equation can be used to eliminate h_2 from (3). We then can solve the conservation-equations formally for

$$\frac{\varrho_1}{\varrho_2} = \frac{v_2}{v_1} = 1 + \frac{(g_2/\gamma\, M_1{}^2) - 1}{g_2 + 1}\{1 \pm \sqrt{1 + \varepsilon}\}, \quad (6)$$

$$\frac{p_2}{p_1} = 1 - \frac{g_2 - \gamma\, M_1{}^2}{g_2 + 1}\{1 \pm \sqrt{1 + \varepsilon}\} \quad (7)$$

where $\quad M_1 = v_1/c_1 = v_1 \cdot \sqrt{\varrho_1/\gamma\, p_1}$,

$$\gamma = c_p/c_v \quad \text{(cold gas)} \quad \text{and}$$

$$\varepsilon = \frac{2\,(g_2 + 1)\,(\gamma - g_2)\,\gamma\, M_1{}^2}{(\gamma - 1)\,(g_2 - \gamma\, M_1{}^2)^2}.$$

The minus sign gives the identity while the plus sign represents the shock solution for supersonic flow. With $M_1 \geqq 5$ (strong shock) and any allowed value of g_2, ε is small as compared with 1, so that the general shock equations become

$$\varrho_1/\varrho_2 = v_2/v_1 = (g_2 - 1)/(g_2 + 1), \quad (8)$$

$$p_2/p_1 = 2\,\gamma\, M_1{}^2/(g_2 + 1) \quad (9)$$

and employing (5) $\quad h_2 = [2\,g_2/(g_2 + 1)^2]\,v_1{}^2.$ (10) For an approximate determination of the end values one can guess g_2 and solve directly for the end values ϱ_2, p_2, v_2, h_2. For an exact calculation, however, g_2 has to be treated as an unknown.

Thermal Equilibrium

The plasma is in local thermal equilibrium if electrons, ions and neutrals each have a MAXWELL velocity distribution characterized by the same temperature $T_e = T_i = T_0 = T$, and if the number of ions can be calculated by the SAHA equation governed by the same temperature $T_{Saha} = T$. For the energy consideration, however, it is not necessary to know, whether the excited states obey a BOLTZMANN distribution, as the correlated energy term is neglected (see below).

If the shock front in the plasma is supposedly in thermal equilibrium and $M > 3$, we have 5 equations: (8) counting twice, (9) and the functions of state $\varrho(p, T)$ and $g(p, T)$. These determine the 5 unknowns ϱ_2, p_2, g_2, T_2 and v_2. Equilibrium values for $\varrho(p, T)$ and $h(p, T)$ and thus

$$g(p, T) = h/(h - p/\varrho)$$

are tabulated for many gases over a large variety of pressures and temperatures [9-13]. Using these ta-

[9] F. BURHORN and R. WIENECKE, LANDOLT-BÖRNSTEIN, Vol. II/4, p. 717.
[10] F. BOSNJAKOVIC, W. SPRINGE, and K. F. KNOCHE, Z. Flugwissenschaft 10, 413 [1962].
[11] K. S. DRELLISHAK, D. P. AESCHLIMAN, and A. B. CAMPBELL, Phys. Fluids 8, 1590 [1965].
[12] R. KUTHE and KL. K. NEUMANN, Ber. Bunsenges. Phys. Chem. 68, 692 [1964].
[13] W. FENEBERG, Zustandswerte von Helium, Institut für Plasmaphysik Garching, unveröffentlicht.

bles the end values can easily be calculated as function of the MACH number.

We select a pressure $p_2 = \tilde{p}_2$ and solve (8) graphically for this pressure $\tilde{p}_2$. This is possible because we can plot both the right side

$$(g_2 - 1)/(g_2 + 1) = [g_2(\tilde{p}_2) - 1]/[g_2(\tilde{p}_2) + 1]$$

and the left side

$$\varrho_1/\varrho_2 = \varrho_1/\varrho(\tilde{p}_2)$$

as function of temperature, taking $\tilde{p}_2$ as a constant parameter, Fig. 4. The intersection of the curves satisfies Eq. (8) for the pressure $\tilde{p}_2$. The abscissa of the intersection point therefore gives the temperature $\tilde{T}_2$ associated with $\tilde{p}_2$ while the ordinate a
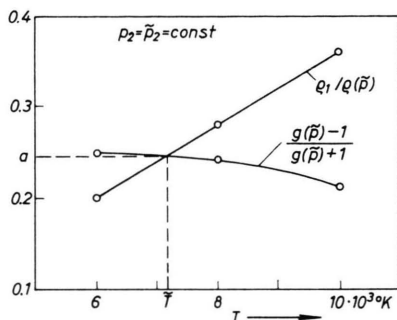
Fig. 4. Graphical solution of Eq. (8).

gives $\tilde{\varrho}_2 = \varrho_1/a$ and $\tilde{g}_2 = (1 + a)/(1 - a)$, as $\tilde{g}_2$ now is known we can solve (9) for $\tilde{M}_1$ thus determining the initial velocity $\tilde{v}_1$ which is required to reach this combination of end values. Finally v_2 can be calculated with (8). These shock front parameters can be plotted as a function of the MACH number. In order to get more points in these graphs, one has to start the calculations with different pressures p_2. It will be noticed that every point obtained with this procedure is an exact value and no iteration is required. Fig. 5, 6, 7 show an example. ϱ_2/ϱ_1, p_2/p_1 and T_2 are given as functions of the MACH number for a strong shock in argon [$h(p_1 T)$ and $\varrho(p, T)$ from [10]]. Parameter is the initial pressure p_1. Compression-ratio, Fig. 5, and temperature, Fig. 6, are very different from the values for a non reacting gas with $\gamma = 5/3 = $ constant, and depend on the initial pressure p_1. The pressure ratio, Fig. 7, does not deviate very much from the inert gas curve ($\gamma = 5/3$), and shows so little variation with the initial pressure, that $p_1 = 10^{-2}$ atm and $p_1 = 10^{-3}$ atm give identical curves in the scale of Fig. 7.
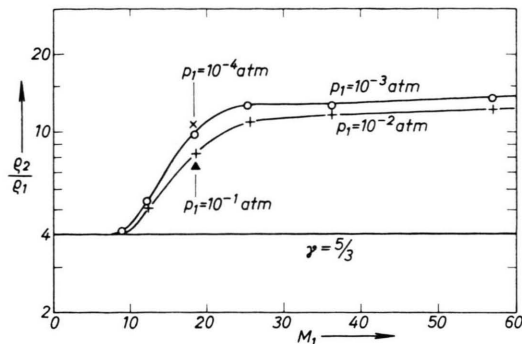
Fig. 5. Compression ratio for a shock wave in argon as function of MACH number, initial pressure p_1 as parameter. $T_1 = 290$ °K, $c = 218$ m/sec. Also shown is the curve for a nonreacting gas with $\gamma = 5/3$.
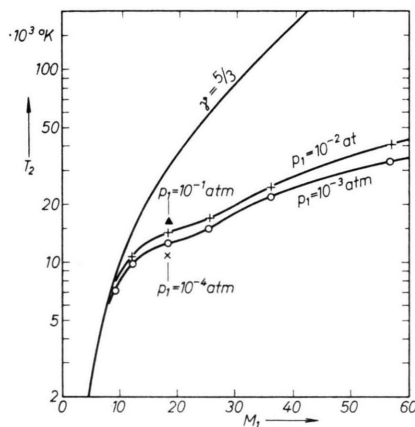
Fig. 6. Shock front temperature in argon as function of the MACH number, initial pressure p_1 as parameter, $T_1 = 290$ °K. Also shown is the curve for a non-reacting gas with $\gamma = 5/3$.
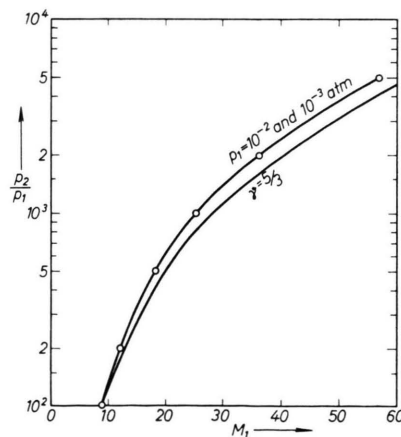
Fig. 7. Pressure ratio for a shock wave in argon as function of MACH number, $T_1 = 290$ °K This curve applies to both, $p_1 = 10^{-3}$ atm and $p_1 = 10^{-2}$ atm, lower curve for nonreacting gas with $\gamma = 5/3$.

Deviations from Thermal Equilibrium

In the absence of thermal equilibrium we have to check whether the equations of our system are still valid, and for this end we have to consider how the properties of state depend on the composition of the plasma. For simplicity we consider a single ionized plasma composed of electrons, ions and neutrals (subscripts e, i, 0). As well known [8], the total pressure p is defined as the sum of the partial pressure

$$p = p_i + p_e + p_0 .$$

The density is given by

$$\varrho = \varrho_i + \varrho_e + \varrho_0 = n_i m_i + n_e m_e + n_0 m_0$$

where n is the number density and m the mass of the particle. The enthalpy

$$h = (\varrho_i/\varrho) h_i + (\varrho_e/\varrho) h_e + (\varrho_0/\varrho) h_0$$

is composed of the partial enthalpies h_i, h_e, h_0 according to the relative abundance of species

$$c_j = n_j m_j / \sum_j n_j m_j .$$

If we assume that ions, electrons and neutrals each have a MAXWELL velocity distribution, so that we may attribute temperatures T_i, T_e, T_0 (which may differ) to each of the species, then the partial enthalpies can be written (E_i = ionization energy) [8]

$$h_i = \tfrac{5}{2} k T_i/m_i + E_i/m_i ,$$
$$h_e = \tfrac{5}{2} k T_e/m_e ,$$
$$h_0 = \tfrac{5}{2} k T_0/m_0 .$$

We have neglected the logarithmic term, which means we neglected the net excitation energy compared with the total thermal and ionization energy. The enthalpy then becomes:

$$h = (1/\varrho) [n_i E_i + \tfrac{5}{2} (n_i k T_i + n_e k T_e + n_0 k T_0)]$$

and with $n_j k T_j = p_j$ we obtain

$$h = (1/\varrho) [n_i E_i + \tfrac{5}{2} p] , \qquad (11)$$

similarly the internal energy is defined by

$$u = (1/\varrho) [n_i E_i + \tfrac{3}{2} p]$$

and thus
$$g = \frac{h}{u} = \frac{n_i E_i + \tfrac{5}{2} p}{n_i E_i + \tfrac{3}{2} p} . \qquad (12)$$

All the variables involved in the three equations of conservation (1), (2) and (3) are defined. Furthermore the thermodynamical relationship $u = h - p/\varrho$ is valid, so that the definition $g = h/u$ is equivalent to stating

$$h = [g/(g-1)] \cdot p/\varrho .$$

Consequently (8), and (9) are valid in the non thermal plasma. However, the only equation of state available is (12), which gives $g(p, n_i)$ and does not include undefined temperature. Thus we have only four equations — (8) counting twice, (9) and (12) — to determine 5 shock-front parameters p_2, ϱ_2, v_2, g_2, n_i. This means nature has freedom to chose a state of non-equilibrium in accordance with the boundary conditions implied by the experiment. In other words, different experiments will produce different forms of non equilibrium. But if we can measure any one of the shock front parameters then all others can be calculated analytically. Suppose, for example, the ratio

$$v_2/v_1 = \varrho_1/\varrho_2 = a$$

was measured and all the initial values (index 1) are known, then we get

$$g_2 = (1+a)/(1-a),$$
$$p_2 = p_1(1-a) \gamma M_1^2$$

and $\qquad n_i = p_1(1-a)(1-4a) \gamma M_1^2/2 a E_i .$

Furthermore, if required, h_2 can be determined from (11). A similar solution is possible, if p_2 or n_i is measured.

It might be of interest to know in which direction the shock front parameters will tend to change, if the plasma deviates from thermal equilibrium. We have to distinguish between an overpopulation $n_i > n_{i,\,\text{Saha}}$ (frozen ionization) and an underpopulation of ions $n_i < n_{i,\,\text{Saha}}$. Differentiating g in (12) with respect to n_i we get

$$\frac{\partial g}{\partial n_i} = -p E_i (\tfrac{3}{2} p + n_i E_i)^{-2} < 0 .$$

The right side is always negative. Thus an overpopulation will reduce the effective g, while an underpopulation of the ions will raise g. If we apply this information to Eqs. (8) and (9), we see that with $n_i > n_{i,\,\text{Saha}}$ the pressure increase p_2/p_1 and the compression ratio ϱ_2/ϱ_1 both will be larger than in thermal equilibrium.